

TOWARDS HILBERT-KUNZ DENSITY FUNCTIONS IN CHARACTERISTIC 0

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ABSTRACT. For a pair (R, I) , where R is a standard graded domain over an algebraically closed field of characteristic 0 and I is a graded ideal with $\ell(R/I) < \infty$, we prove that, as $p \rightarrow \infty$, the convergence of the HK density function $f^p(R_p, I_p)$ (in L^∞ norm) is equivalent to the convergence of $f_m^p(R_p, I_p)$, for any fixed $m \geq \dim \operatorname{Proj} R$. This implies that the existence of $\lim_{p \rightarrow \infty} e_{HK}(R_p, I_p)$ is equivalent to the existence of $\lim_{p \rightarrow \infty} \ell(R_p/I_p^{[p^m]})/p^{md}$, for any such fixed m .

In particular, in char 0, to define the HK density function $f(R, I)$ (HK multiplicity $e_{HK}(R, I)$) it is enough to prove the existence of $\lim_{p \rightarrow \infty} f_m^p(R_p, I_p)$ ($\lim_{p \rightarrow \infty} \ell(R_p/I_p^{[p^m]})/p^{md}$ respectively), for any fixed $m \geq \dim \operatorname{Proj} R$.

1. INTRODUCTION

Let R be a Noetherian ring of prime characteristic $p > 0$ and of dimension d and let $I \subseteq R$ be an ideal of finite colength. Then we recall that the Hilbert-Kunz multiplicity of R with respect to I is defined as

$$e_{HK}(R, I) = \lim_{n \rightarrow \infty} \frac{\ell(R/I^{[p^n]})}{p^{nd}},$$

where $I^{[p^n]}$ = n -th Frobenius power of I = ideal generated by p^n -th power of elements of I . This is an ideal of finite colength and $\ell(R/I^{[p^n]})$ denotes the length of the R -module $R/I^{[p^n]}$. This invariant had been introduced by E. Kunz and existence of the limit was proved by Monsky [M]. It carries information about char p related properties of the ring, but at the same time is difficult to compute (even in the graded case) as various standard techniques, used for studying multiplicities, are not applicable for the invariant e_{HK} . It is natural to ask if the notion of this invariant can be extended to the ‘char 0’ case. A natural way to attempt this for a pair (R, I) (from now onwards, unless stated otherwise, by a pair (R, I) , we mean R is a standard graded ring and $I \subset R$ is a graded ideal of finite colength) could be as follows: Suppose R is a finitely generated algebra and a domain over a field k of characteristic 0 and $I \subseteq R$ is an ideal of finite colength. Let (A, R_A, I_A) be a spread of the pair (R, I) (see Definition 3.2), where $A \subset k$ is a finitely generated algebra over \mathbb{Z} . Then we may (tentatively) define

$$e_{HK}^\infty(R, I) := \lim_{s \rightarrow s_0} e_{HK}(R_s, I_s),$$

where $R_s = R_A \otimes_A \bar{k}(s)$ and $I_s = I_A \otimes_A \bar{k}(s)$, s_0 = the generic point of $\operatorname{Spec} A$ and $s \in$ closed points of $\operatorname{Spec} A$. Or consider a simpler situation: R is a finitely generated \mathbb{Z} -algebra and a domain, $I \subset R$ such that R/I is an abelian group of finite rank then let

$$e_{HK}^\infty(R, I) := \lim_{p \rightarrow \infty} e_{HK}(R_p, I_p), \quad \text{where } R_p = R \otimes_{\mathbb{Z}} \mathbb{Z}/p\mathbb{Z} \quad \text{and} \quad I_p = I \otimes_{\mathbb{Z}} \mathbb{Z}/p\mathbb{Z}.$$

In case of dimension $R = 1$, we know that the Hilbert-Kunz multiplicity coincides with the Hilbert-Samuel multiplicity hence it is independent of p , for large p .

For homogeneous coordinate rings of plane curves, with respect to the maximal graded ideal (in [T1], [M]), nonsingular curves with respect to a graded ideal I (in [T2]), diagonal hypersurfaces (in [GM]), it has been shown that $e_{HK}(R_p, I_p)$ varies with p and the limit exists.

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Moreover using the above mentioned result of [T2] and Proposition 2.17 of [T4], it can be proved that, for a Segre product of any finite number of projective curves, such a limit exists. Then there are other cases, where $e_{HK}(R_p, I_p)$ is independent of p and therefore the limit exists.

Since

$$e_{HK}^\infty(R, I) := \lim_{p \rightarrow \infty} \lim_{n \rightarrow \infty} \frac{\ell(R_p/I_p^{[p^n]})}{(p^n)^d},$$

it seems harder to compute as such, as $\lim_{n \rightarrow \infty} \ell(R_p/I_p^{[p^n]})/(p^n)^d$ itself does not seem easily computable (even in the graded case). To make this invariant more approachable the following question was posed in [BLM]:

Question. Suppose $e_{HK}^\infty(R, I)$ exists, is it true that for any fixed $n \geq 1$

$$e_{HK}^\infty(R, I) = \lim_{p \rightarrow \infty} \frac{\ell(R_p/I_p^{[p^n]})}{(p^n)^d}?$$

The main result of their paper was to give an affirmative answer in the case of a 2 dimensional standard graded normal domain R with respect to a homogeneous ideal I . Note that the existence of $e_{HK}^\infty(R, I)$, in this case, was proved earlier in [T2].

The proof goes as follows: Recall that for a vector bundle V on a smooth (projective and polarized) variety, we have the well defined *HN data*, namely $\{r_i(V), \mu_i(V)\}_i$, where $r_i(V) = \text{rank}(E_i/E_{i-1})$ and $\mu_i(V) = \text{slope of } E_i/E_{i-1}$ and

$$0 \subset E_1 \subset E_2 \subset \cdots \subset E_l \subset V$$

is the Harder-Narasimhan filtration of V .

Let $X_p = \text{Proj } R_p$, which is a nonsingular projective curve, and let I_p be generated by homogeneous elements of degrees d_1, \dots, d_μ then we have the vector bundle V_p on X_p given by the following canonical exact sequence of \mathcal{O}_{X_p} -modules

$$0 \longrightarrow V_p \longrightarrow \oplus_i \mathcal{O}_{X_p}(1 - d_i) \longrightarrow \mathcal{O}_{X_p}(1) \longrightarrow 0.$$

Then, by Proposition 1.16 in [T2], there is a constant determined by genus of X_p and rank V_p , such that for $s \geq 1$

$$(1.1) \quad \left| \sum_j r_j(F^{s*}V_p) \mu_j(F^{s*}V_p)^2 - \sum_i r_i(V_p) \mu_i(V_p)^2 \right| \leq C/p.$$

(Here F is the absolute Frobenius morphism, and F^s is the s -fold iterate.) Note that the HN filtration and hence the HN data of V_p stabilizes for $p \gg 0$ ([Mar]).

On the other hand, the methods used in [B] and [T1] (relating e_{HK} and the HN data of $F^{s*}V$) imply that, for $s \geq 1$,

$$\frac{1}{(p^s)^2} \ell \left(\frac{R_p}{I_p^{[p^s]}} \right) = \frac{\deg X_p}{2} \left(\sum_j r_j(F^{s*}V_p) \mu_j(F^{s*}V_p)^2 \right) - \frac{\deg X_p}{2} \left(\sum_i d_i^2 \right) + O(1/p).$$

Hence, for any $s \geq 1$, by Equation (1.1)

$$\frac{1}{(p^s)^2} \ell \left(\frac{R_p}{I_p^{[p^s]}} \right) = \frac{\deg X_p}{2} \left(\sum_i r_i(V_p) \mu_i(V_p)^2 \right) - \frac{\deg X_p}{2} \left(\sum_i d_i^2 \right) + O(1/p),$$

which implies

$$e_{HK}(R_p, I_p) = \frac{\deg X_p}{2} \left(\sum_i r_i(V_p) \mu_i(V_p)^2 \right) - \frac{\deg X_p}{2} \left(\sum_i d_i^2 \right) + O(1/p).$$

Also, here the ‘error term’ $|O(1/p)| \leq C_2/p$, where C_2 depends on the invariants like genus of X_p and rank V_p . Hence

$$\lim_{p \rightarrow \infty} e_{HK}(R_p, I_p) = \lim_{p \rightarrow \infty} \frac{1}{(p^s)^2} \ell \left(\frac{R_p}{I_p^{[p^s]}} \right), \quad \text{for any } s \geq 1.$$

Thus here

- (1) one relates $\ell(R_p/I_p^{[p^s]})$ with the HN data of $F^{s*}V_p$, for $s \geq 1$ ([B] and [T1]).
- (2) The HN data of $F^{s*}V_p$ is related to the HN data of V_p ([T2]).
- (3) The restriction of the relative HN filtration of V_A on X_A (where V_A is a spread of V_0 in char 0) remains the HN filtration of V_p for large p ([Mar]).

In particular for a pair (R, I) , where $\text{char } R = p > 0$, with the associated syzygy bundle V (as above), the proof uses the comparison of $\ell(R/I^{[p^s]})$ with the HN data of the syzygy bundle V and the other well behaved invariants of (R, I) (which have well defined notion in all characteristics and are well behaved vis-a-vis reduction mod p).

However note that (3) is valid for $\dim R \geq 2$, and (2) also holds for $\dim R \geq 3$ (proved relatively recently in [T3]). But (1) does not seem to hold in higher dimension, due to the existence of cohomologies other than $H^0(-)$ and $H^1(-)$ (therefore one can not use anymore the semistability property of a vector bundle to compute h^0 of almost all its twists, by a very ample line bundle).

In this paper, we approach the problem by comparing $\frac{1}{(p^n)^d} \ell(R/I^{[p^n]})$ and $\frac{1}{(p^{n+1})^d} \ell(R/I^{[p^{n+1}]})$ directly, for $n \geq 1$, taking into account that both are graded. For this we phrase the problem in a more general setting: By the theory of Hilbert-Kunz *density function* (which was introduced and developed in [T4]), for a pair (R, I) where R is a domain of char $p > 0$, there exists a sequence of functions $\{f_n^p : [0, \infty) \rightarrow \mathbb{R}\}_n$ such that

$$\frac{1}{(p^n)^d} \ell(R/I^{[p^n]}) = \int_0^\infty f_n^p(x) dx \quad \text{and} \quad \lim_{n \rightarrow \infty} \frac{1}{(p^n)^d} \ell(R/I^{[p^n]}) = \int_0^\infty f^p(x) dx,$$

where the map $f^p : [0, \infty) \rightarrow \mathbb{R}$ is given by $f^p(x) = \lim_{n \rightarrow \infty} f_n^p(x)$ is called the HK density function of (R_p, I_p) (the existence and properties of the limit defining f^p are proved in [T4]). We show here that, for all $x \in [1, \infty)$,

$$\text{if } \lim_{p \rightarrow \infty} f^p(x) := \lim_{p \rightarrow \infty} \lim_{n \rightarrow \infty} f_n^p(x) \quad \text{exists then} \quad \lim_{p \rightarrow \infty} f^p(x) = \lim_{p \rightarrow \infty} f_m^p(x),$$

for any fixed $m \geq d - 1$, where $d - 1 = \dim \text{Proj } R$.

The main point (Proposition 2.11) is to give a bound on the difference $\|f_n^p - f_{n+1}^p\|$, in terms of a power of p and invariants which are well behaved under reduction mod p , where $\|g\| := \sup\{g(x) \mid x \in [1, \infty)\}$ is the L^∞ norm. Since the union of the support of all f_n is contained in a compact interval, a similar bound (Corollary 2.12) holds for the difference $|\ell(R/I^{[p^n]})/(p^n)^d - \ell(R/I^{[p^{n+1}]})/(p^{n+1})^d|$. More precisely we prove the following

Theorem 1.1. *Let R be a standard graded domain of dimension $d \geq 2$, over an algebraically closed field k of characteristic 0. Let $I \subset R$ be a homogeneous ideal of finite colength. Let (A, R_A, I_A) be a spread (see Definition 3.2). Then, for a closed point $s \in \text{Spec}(A)$, let the function*

$$f_n^s(x) : [1, \infty) \rightarrow [0, \infty) \quad \text{be given by} \quad f_n^s(x) = \frac{1}{q^{d-1}} \ell \left(\frac{R_s}{I_s^{[q]}} \right)_{[xq]}.$$

Let the HK density function of (R_s, I_s) be given by

$$f^s(x) = \lim_{n \rightarrow \infty} f_n^s(x).$$

Let $s_0 \in \text{Spec } Q(A)$ denote the generic point of $\text{Spec}(A)$. Then

- (1) *there exists a constant C (given in terms of invariants of (R_{s_0}, I_{s_0})) and an open dense subset $\text{Spec}(A')$ of $\text{Spec}(A)$ such that for every closed point $s \in \text{Spec}(A')$ and $n \geq 1$,*

$$\|f_n^s - f_{n+1}^s\| < C/p^{n-d+2},$$

where $p = \text{char } k(s)$. In particular, for any $m \geq d - 1$,

$$\lim_{s \rightarrow s_0} \|f_m^s - f^s\| = 0.$$

- (2) There exists a constant C_1 (given in terms of invariants of (R_{s_0}, I_{s_0})) and an open dense subset $\text{Spec}(A')$ of $\text{Spec}(A)$, such that for every closed point $s \in \text{Spec}(A')$ and $n \geq 1$, we have

$$\left| \frac{1}{p^{nd}} \ell \left(\frac{R_s}{I_s^{[p^n]}} \right) - \frac{1}{p^{(n+1)d}} \ell \left(\frac{R_s}{I_s^{[p^{n+1}]}} \right) \right| \leq \frac{C_1}{p^{n-d+2}}.$$

- (3) For any $m \geq d - 1$,

$$\lim_{s \rightarrow s_0} \left[\frac{1}{p^{md}} \ell \left(\frac{R_s}{I_s^{[p^m]}} \right) - e_{HK}(R_s, I_s) \right] = 0.$$

As a result we have

Corollary 1.2. *Let R be a standard graded domain and a finitely generated \mathbb{Z} -algebra, let $I \subset R$ be a homogeneous ideal of finite colength. such that for almost all p , the fiber over p , $R_p := R \otimes_{\mathbb{Z}} \mathbb{Z}/p\mathbb{Z}$ is a standard graded ring of dimension d , which is geometrically integral, and $I_p \subset R_p$ is a homogenous ideal of finite colength. Then*

- (1) *there exists a constant C_1 given in terms of invariants of R and I such that, for $n \geq 1$, we have*

$$\left| \frac{1}{p^{nd}} \ell \left(\frac{R_p}{I_p^{[p^n]}} \right) - \frac{1}{p^{(n+1)d}} \ell \left(\frac{R_p}{I_p^{[p^{n+1}]}} \right) \right| \leq \frac{C_1}{p^{n-d+2}}.$$

- (2) *For any fixed $m \geq d - 1$,*

$$\lim_{p \rightarrow \infty} \left[e_{HK}(R_p, I_p) - \frac{1}{p^{md}} \ell \left(\frac{R_p}{I_p^{[p^m]}} \right) \right] = 0.$$

In particular, for any fixed $m \geq d - 1$,

$$e_{HK}^{\infty}(R, I) := \lim_{p \rightarrow \infty} e_{HK}(R_p, I_p) \text{ exists} \iff \lim_{p \rightarrow \infty} \frac{1}{p^{md}} \ell \left(\frac{R_p}{I_p^{[p^m]}} \right) \text{ exists}.$$

In particular the last assertion of the above corollary answers the above mentioned question of [BLM] affirmatively, for all (R, I) , where R is a standard graded domain and $I \subset R$ is a graded ideal of finite colength.

Moreover the proof, even in the case of dimension 2 (unlike the proof in [BLM]) does not rely on earlier results of [B], [T1] and [T2]. In particular, since we do not use Harder-Narasimhan filtrations, we do not need a normality hypothesis on the ring R .

Remark 1.3. If $e_{HK}^{\infty}(R, I)$ exists for a pair (R, I) , whenever R is a standard graded domain, defined over an algebraically closed field of characteristic 0, then one can check that $e_{HK}^{\infty}(R, I)$ exists for any pair (R, I) where R is a standard graded ring over a field k of characteristic 0: Let $\bar{R} = R \otimes_k \bar{k}$. Let $\{q_1, \dots, q_r\} = \{q \in \text{Ass}(\bar{R}) \mid \dim \bar{R}/q = \dim R\}$ then we have a spread $(A, \bar{R}_A, \bar{I}_A)$ of (\bar{R}, \bar{I}) such that $\{q_{1s}, \dots, q_{rs}\} = \{q_s \in \text{Ass}(\bar{R}_s) \mid \dim \bar{R}_s/q_s = \dim \bar{R}_s\}$ and, for each i , $\ell((\bar{R}_s)_{q_{is}}) = l_i$, a constant independent of s . This implies that

$$e_{HK}(\bar{R}_s, \bar{I}_s) = \sum_{i=1}^r l_i e_{HK} \left(\frac{\bar{R}_s}{q_{is}}, \frac{\bar{I}_s + q_{is}}{q_{is}} \right),$$

which implies

$$\lim_{s \rightarrow s_0} e_{HK}(\bar{R}_s, \bar{I}_s) = \sum_{i=1}^r l_i \lim_{s \rightarrow s_0} e_{HK} \left(\frac{\bar{R}_s}{q_{is}}, \frac{\bar{I}_s + q_{is}}{q_{is}} \right) = \sum_{i=1}^r l_i e_{HK}^{\infty} \left(\frac{\bar{R}}{q_i}, \frac{\bar{I} + q_i}{q_i} \right).$$

Hence, in this situation, one can define

$$e_{HK}^\infty(R, I) := e_{HK}^\infty(\bar{R}, \bar{I}) = \sum_{i=1}^r l_i e_{HK}^\infty\left(\frac{\bar{R}}{q_i}, \frac{\bar{I} + q_i}{q_i}\right).$$

2. A KEY PROPOSITION

Throughout this section, R is a Noetherian standard graded integral domain of dimension d over an algebraically closed field k of char $p > 0$, I is a homogeneous ideal of R such that $\ell(R/I) < \infty$. Let h_1, \dots, h_μ be a set of homogeneous generators of I of degrees d_1, \dots, d_μ respectively.

Let $X = \text{Proj } R$; then we have an associated canonical short exact sequence of locally free sheaves of \mathcal{O}_X -modules (moreover the sequence is locally split exact)

$$(2.1) \quad 0 \longrightarrow V \longrightarrow \oplus_i \mathcal{O}_X(1 - d_i) \longrightarrow \mathcal{O}_X(1) \longrightarrow 0,$$

where $\mathcal{O}_X(1 - d_i) \longrightarrow \mathcal{O}_X(1)$ is given by the multiplication by the element h_i .

For a coherent sheaf \mathcal{Q} of \mathcal{O}_X -modules, the sequence of \mathcal{O}_X -modules

$$0 \longrightarrow F^{n*}V \otimes \mathcal{Q}(m) \longrightarrow \oplus_i \mathcal{Q}(q - qd_i + m) \longrightarrow \mathcal{Q}(q + m) \longrightarrow 0$$

is exact as the short exact sequence (2.1) is locally split as \mathcal{O}_X -modules (as usual, $q = p^n$ and F^n is the n^{th} iterate of the absolute Frobenius morphism). Therefore we have a long exact sequence of cohomologies

$$(2.2) \quad 0 \longrightarrow H^0(X, F^{n*}V \otimes \mathcal{Q}(m)) \longrightarrow \oplus_i H^0(X, \mathcal{Q}(q - qd_i + m)) \xrightarrow{\phi_{m,q}(\mathcal{Q})} H^0(X, \mathcal{Q}(q + m)) \\ \longrightarrow H^1(X, F^{n*}V \otimes \mathcal{Q}(m)) \longrightarrow \dots,$$

for $m \geq 0$ and $q = p^n$.

Definition 2.1. Let \mathcal{Q} be a coherent sheaf of \mathcal{O}_X -modules and let $\mathcal{O}_X(1)$ be a very ample line bundle on X . We say that \mathcal{Q} is \tilde{m} -regular (or \tilde{m} is a regularity number of \mathcal{Q}) with respect to $\mathcal{O}_X(1)$, if for all $m \geq \tilde{m}$

- (1) the canonical multiplication map $H^0(X, \mathcal{Q}(m)) \otimes H^0(X, \mathcal{O}_X(1)) \longrightarrow H^0(X, \mathcal{Q}(m+1))$ is surjective and
- (2) $H^i(X, \mathcal{Q}(m-i)) = 0$, for $i \geq 1$.

Notations 2.2. Let

$$P_{(R, \mathbf{m})}(m) = \tilde{e}_0 \binom{m+d-1}{d} - \tilde{e}_1 \binom{m+d-2}{d-1} + \dots + (-1)^d \tilde{e}_d$$

be the Hilbert-Samuel polynomial of R with respect to the graded maximal ideal \mathbf{m} . Therefore

$$\chi(X, \mathcal{O}_X(m)) = \tilde{e}_0 \binom{m+d-1}{d-1} - \tilde{e}_1 \binom{m+d-2}{d-2} + \dots + (-1)^{d-1} \tilde{e}_{d-1}.$$

Let \bar{m} be a positive integer such that

- (1) \bar{m} is a regularity number for $(X, \mathcal{O}_X(1))$, and
- (2) $R_m = h^0(X, \mathcal{O}_X(m))$, for all $m \geq \bar{m}$. In particular $\ell(R/\mathbf{m}^m) = P_{(R, \mathbf{m})}(m)$, for all $m \geq \bar{m}$.

Let $l_1 = h^0(X, \mathcal{O}_X(1))$ and let $n_0 \geq 1$ be an integer such that $R_{n_0} \subseteq I$.

We also denote $\dim_k \text{Coker } \phi_{m,q}(\mathcal{Q})$ by $\text{coker } \phi_{m,q}(\mathcal{Q})$.

Remark 2.3. (1) The canonical map $\oplus_m R_m \longrightarrow \oplus_m H^0(X, \mathcal{O}_X(m))$ is injective, as R is an integral domain.

- (2) For $m + q \geq m_R(q) = \bar{m} + n_0(\sum_i d_i)q$, we have $\text{coker } \phi_{m,q}(\mathcal{O}_X) = \ell(R/I^{[q]})_{m+q} = 0$: Because $m_R(q) = \bar{m} + n_0\mu q + n_0(\sum_i (d_i - 1))q \implies q - qd_i + m \geq \bar{m}$, for all i . Hence the map $\phi_{m,q}(\mathcal{O}_X)$ is the map $\oplus_i R_{q-qd_i+m} \longrightarrow R_{m+q}$, where the map $R_{q-qd_i+m} \rightarrow R_{m+q}$ is given by multiplication by the element h_i^q . Therefore, $\text{coker } \phi_{m,q}(\mathcal{O}_X) = \ell(R/I^{[q]})_{m+q}$. Moreover, by Lemma 2.10, we have $\ell(R/I^{[q]})_{m+q} = 0$, as $m + q \geq \bar{m} + n_0\mu q$.
- (3) For $C_R = (\mu)h^0(X, \mathcal{O}_X(\bar{m}))$, we have

$$(2.3) \quad |\text{coker } \phi_{m,q}(\mathcal{O}_X) - \ell(R/I^{[q]})_{m+q}| \leq C_R,$$

for all $n, m \geq 0$ and $q = p^n$: Because

if $m + q < \bar{m}$, then

$$|\text{coker } \phi_{m,q}(\mathcal{O}_X) - \ell(R/I^{[q]})_{m+q}| \leq h^0(X, \mathcal{O}_X(m+q)) \leq h^0(X, \mathcal{O}_X(\bar{m})).$$

If $m + q \geq \bar{m}$, then $h^0(X, \mathcal{O}_X(m+q)) = \ell(R_{m+q})$ and therefore

$$|\text{coker } \phi_{m,q}(\mathcal{O}_X) - \ell(R/I^{[q]})_{m+q}| \leq \sum_i^\mu |h^0(X, \mathcal{O}_X(q - qd_i + m)) - \ell(R_{q-qd_i+m})|.$$

Now, if $q - qd_i + m < \bar{m}$ then $\ell(R_{q-qd_i+m}) \leq h^0(X, \mathcal{O}_X(q - qd_i + m)) \leq h^0(X, \mathcal{O}_X(\bar{m}))$, and if $q - qd_i + m \geq \bar{m}$ then $R_{q-qd_i+m} = H^0(X, \mathcal{O}_X(q - qd_i + m))$.

Hence

$$|\text{coker } \phi_{m,q}(\mathcal{O}_X) - \ell(R/I^{[q]})_{m+q}| \leq \mu h^0(X, \mathcal{O}_X(\bar{m})).$$

Definition 2.4. Let \mathcal{Q} be a coherent sheaf of \mathcal{O}_X -modules of dimension \bar{d} and let $\tilde{m} \geq 1$ be the least integer which is a regularity number for \mathcal{Q} with respect to $\mathcal{O}_X(1)$. Then we define $C_0(\mathcal{Q})$ and $D_0(\mathcal{Q})$ as follows: Let $a_1, \dots, a_{\bar{d}} \in H^0(X, \mathcal{O}_X(1))$ be such that we have a short exact sequence of \mathcal{O}_X -modules

$$0 \rightarrow \mathcal{Q}_i(-1) \xrightarrow{a_i} \mathcal{Q}_i \rightarrow \mathcal{Q}_{i-1} \rightarrow 0, \quad \text{for } 0 < i \leq \bar{d},$$

where $\mathcal{Q}_d = \mathcal{Q}$ and $\mathcal{Q}_i = \mathcal{Q}/(a_{\bar{d}}, \dots, a_{i+1})\mathcal{Q}$, for $0 \leq i < \bar{d}$, with $\dim \mathcal{Q}_i = i$. We define

$$C_0(\mathcal{Q}) = \min\left\{\sum_{i=0}^{\bar{d}} h^0(X, \mathcal{Q}_i) \mid a_1, \dots, a_{\bar{d}} \text{ is a } \mathcal{Q}\text{-sequence as above}\right\},$$

$$D_0(\mathcal{Q}) = h^0(X, \mathcal{Q}(\tilde{m})) + 2(\bar{d} + 1)(\max\{q_0, q_1, \dots, q_{\bar{d}}\}),$$

where

$$\chi(X, \mathcal{Q}(m)) = q_0 \binom{m + \bar{d}}{\bar{d}} - q_1 \binom{m + \bar{d} - 1}{\bar{d} - 1} + \dots + (-1)^{\bar{d}} q_{\bar{d}}$$

is the Hilbert polynomial of \mathcal{Q} .

Lemma 2.5. Let \mathcal{Q} be a coherent sheaf of \mathcal{O}_X -modules of dimension \bar{d} . Let P be a locally-free sheaf of \mathcal{O}_X -modules which fits into a short exact sequence of locally-free sheaves of \mathcal{O}_X -modules

$$(2.4) \quad 0 \longrightarrow P \longrightarrow \oplus_i \mathcal{O}_X(-b_i) \longrightarrow P'' \longrightarrow 0, \text{ where } b_i \geq 0.$$

Then, for $\tilde{\mu} = \text{rk}(P) + \text{rk}(P'')$ and, for all $n, m \geq 0$, we have

$$h^0(X, \mathcal{Q}(m+q)) \leq D_0(\mathcal{Q})(m+q)^{\bar{d}} \quad \text{and} \quad h^0(F^{n*}P \otimes \mathcal{Q}(m)) \leq (\tilde{\mu})C_0(\mathcal{Q})(m^{\bar{d}} + 1).$$

Proof. Let \tilde{m} be a regularity number for \mathcal{Q} , then by Definition 2.4, we have

$$h^0(X, \mathcal{Q}(m+q)) \leq D_0(\mathcal{Q})(m+q)^{\bar{d}}, \quad \text{for all } n, m \geq 0.$$

Let $\mathcal{Q}_{\bar{d}} = \mathcal{Q}$. Let $a_{\bar{d}}, \dots, a_1 \in H^0(X, \mathcal{O}_X(1))$ with the exact sequence of \mathcal{O}_X -modules

$$0 \longrightarrow \mathcal{Q}_i(-1) \xrightarrow{a_i} \mathcal{Q}_i \longrightarrow \mathcal{Q}_{i-1} \longrightarrow 0,$$

where $\mathcal{Q}_i = \mathcal{Q}_{\bar{d}}/(a_{\bar{d}}, \dots, a_{i+1})\mathcal{Q}_{\bar{d}}$, for $0 \leq i \leq \bar{d}$, and realizing the minimal value $C_0(\mathcal{Q})$. Now, by the exact sequence (2.4), we have the following short exact sequence of \mathcal{O}_X -sheaves

$$0 \longrightarrow F^{n*}P \otimes \mathcal{Q}_i \longrightarrow \oplus_j \mathcal{Q}_i(-qb_j) \longrightarrow F^{n*}P'' \otimes \mathcal{Q}_i \longrightarrow 0.$$

This implies $H^0(X, F^{n*}P \otimes \mathcal{Q}_i) \hookrightarrow \oplus_j H^0(X, \mathcal{Q}_i(-qb_j))$. Therefore

$$(2.5) \quad h^0(X, F^{n*}P \otimes \mathcal{Q}_i) \leq \sum_j h^0(X, \mathcal{Q}_i(-qb_j)) \leq (\tilde{\mu})h^0(X, \mathcal{Q}_i),$$

as $-b_j \leq 0$. Since $F^{n*}P$ is a locally-free sheaf of \mathcal{O}_X -modules, we have

$$0 \longrightarrow F^{n*}P \otimes \mathcal{Q}_i(m-1) \xrightarrow{a_i} F^{n*}P \otimes \mathcal{Q}_i(m) \longrightarrow F^{n*}P \otimes \mathcal{Q}_{i-1}(m) \longrightarrow 0,$$

which is a short exact sequence of \mathcal{O}_X -sheaves. Now by induction on i , we prove that, for $m \geq 1$,

$$h^0(X, F^{n*}P \otimes \mathcal{Q}_i(m)) \leq (\tilde{\mu}) [h^0(X, \mathcal{Q}_i) + \cdots + h^0(X, \mathcal{Q}_0)] (m^i).$$

For $i = 0$, the inequality holds as $h^0(X, F^{n*}P \otimes \mathcal{Q}_0(m)) \leq (\tilde{\mu})h^0(X, \mathcal{Q}_0)$ (as $\dim \mathcal{Q}_0 = 0$).

Now, for $m \geq 1$, by the inequality 2.5 and by induction on i , we have

$$\begin{aligned} h^0(X, F^{n*}P \otimes \mathcal{Q}_i(m)) &\leq h^0(X, F^{n*}P \otimes \mathcal{Q}_i) + h^0(X, F^{n*}P \otimes \mathcal{Q}_{i-1}(1)) + \cdots + h^0(X, F^{n*}P \otimes \mathcal{Q}_{i-1}(m)) \\ &\leq (\tilde{\mu})h^0(X, \mathcal{Q}_i) + \tilde{\mu} [h^0(X, \mathcal{Q}_{i-1}) + \cdots + h^0(X, \mathcal{Q}_0)] (1 + 2^{i-1} + \cdots + m^{i-1}) \\ &\leq (\tilde{\mu}) [h^0(X, \mathcal{Q}_i) + \cdots + h^0(X, \mathcal{Q}_0)] m^i. \end{aligned}$$

This implies

$$h^0(X, F^{n*}P \otimes \mathcal{Q}(m)) = h^0(X, F^{n*}P \otimes \mathcal{Q}_{\bar{d}}(m)) \leq \tilde{\mu} C_0(\mathcal{Q}) m^{\bar{d}},$$

for all $m \geq 1$. Therefore, for all $0 \leq i \leq \bar{d}$, $h^0(X, F^{n*}P \otimes \mathcal{Q}(m)) \leq \tilde{\mu} C_0(\mathcal{Q}) (m^{\bar{d}} + 1)$, for all $m \geq 0$. This proves the lemma. \square

Lemma 2.6. *There exists a short exact sequence of coherent sheaves of \mathcal{O}_X -modules*

$$0 \longrightarrow \oplus^{p^{d-1}} \mathcal{O}_X(-d) \longrightarrow F_* \mathcal{O}_X \longrightarrow \mathcal{Q} \longrightarrow 0,$$

where \mathcal{Q} is a coherent sheaf of \mathcal{O}_X -modules such that $\dim \operatorname{supp}(\mathcal{Q}) < d - 1$.

Proof. Note that $X = \operatorname{Proj} R$, where $R = \oplus_{n \geq 0} R_n$, is a standard graded domain such that R_0 is an algebraically closed field. Therefore there exists a Noether normalization

$$k[X_0, \dots, X_{d-1}] \longrightarrow R,$$

which is an injective, finite separable graded map of degree 0 (as k is an algebraically closed field). This induces a finite separable affine map $\pi : X \longrightarrow \mathbb{P}_k^{d-1} = S$.

Note that there is also an isomorphism

$$\eta : \mathcal{O}_S^{\oplus n_0} \oplus \mathcal{O}_S(-1)^{\oplus n_1} \oplus \cdots \oplus \mathcal{O}_S(-d+1)^{\oplus n_{d-1}} \longrightarrow F_* \mathcal{O}_S$$

of \mathcal{O}_S -modules, where $\sum n_i = p^{d-1}$.

Now the isomorphism of η implies that the map $\pi^*(\eta) : \oplus_{i=0}^{d-1} \mathcal{O}_X(-i)^{\oplus n_i} \longrightarrow \pi^* F_* \mathcal{O}_S$ is an isomorphism of \mathcal{O}_X -sheaves. Since $0 \leq i \leq d-1$, we also have an injective and generically isomorphic map of \mathcal{O}_X -sheaves

$$\oplus^{p^{d-1}} \mathcal{O}_X(-d) \longrightarrow \oplus_{i=0}^{d-1} \mathcal{O}_X(-i)^{\oplus n_i}.$$

Composing this map with $\pi^*(\eta)$ gives an injective and generically isomorphic map of \mathcal{O}_X -sheaves

$$\alpha : \oplus^{p^{d-1}} \mathcal{O}_X(-d) \longrightarrow \pi^* F_* \mathcal{O}_S.$$

Since π is separable, there is a canonical map $\beta : \pi^* F_* \mathcal{O}_S \longrightarrow F_* \mathcal{O}_X$, of sheaves of \mathcal{O}_X -modules, which is generically isomorphic.

Now we have the composite map

$$\beta \circ \alpha : \oplus^{p^{d-1}} \mathcal{O}_X(-d) \longrightarrow \pi^* F_* \mathcal{O}_S \rightarrow F_* \mathcal{O}_X$$

which is generically an isomorphism. Hence $\dim \operatorname{Coker}(\beta \circ \alpha) < \dim X = d - 1$ and the map $\beta \circ \alpha : \oplus^{p^{d-1}} \mathcal{O}_X(-d) \longrightarrow F_* \mathcal{O}_X$ is injective, as X is an integral scheme. This proves the lemma. \square

Lemma 2.7. *Let*

$$0 \longrightarrow \oplus^{p^{d-1}} \mathcal{O}_X(-d) \longrightarrow F_* \mathcal{O}_X \longrightarrow \mathcal{Q} \longrightarrow 0$$

as in the Lemma 2.6. Then

- (1) \mathcal{Q} is \tilde{m} -regular, where $\tilde{m} = \max\{\bar{m} + d, l_1 - 1\}$, where \bar{m} and l_1 are as given in Notations 2.2.
- (2) For a given d , there exists a universal polynomial function $\bar{P}_1^d(X_0, \dots, X_{d-1}, Y)$ with rational coefficients (and hence independent of p) such that

$$2C_0(\mathcal{Q}) + D_0(\mathcal{Q}) \leq p^{d-1} \bar{P}_1^d(\tilde{e}_0, \tilde{e}_1, \dots, \tilde{e}_{d-1}, \tilde{m}).$$

Proof. (1) The above short exact sequence of \mathcal{O}_X -sheaves gives a long exact sequence of cohomologies

$$\oplus^{p^{d-1}} H^i(X, \mathcal{O}_X(m-d)) \longrightarrow H^i(X, \mathcal{O}_X(mp)) \longrightarrow H^i(X, \mathcal{Q}(m)) \longrightarrow \oplus^{p^{d-1}} H^{i+1}(X, \mathcal{O}_X(m-d)).$$

But $h^i(X, \mathcal{O}_X(m-d-i)) = 0$, for all $m \geq \bar{m} + d$ and $i \geq 1$, which implies that if $m \geq \bar{m} + d$ then $h^i(X, \mathcal{Q}(m-i)) = 0$, for $i \geq 1$, and the map

$$f_{1,m} : H^0(X, (F_* \mathcal{O}_X)(m)) \longrightarrow H^0(X, \mathcal{Q}(m))$$

is surjective. Moreover the canonical map

$$H^0(X, (F_* \mathcal{O}_X)(m)) \otimes H^0(X, \mathcal{O}_X(1)) \longrightarrow H^0(X, (F_* \mathcal{O}_X)(m+1))$$

is

$$f_{2,m} : H^0(X, \mathcal{O}_X(mp)) \otimes H^0(X, \mathcal{O}_X(1))^{[p]} \longrightarrow H^0(X, \mathcal{O}_X(mp+p)),$$

is surjective for $m \geq \tilde{m}$ because it fits into the following canonical diagram

$$\begin{array}{ccc} R_{mp} \otimes R_1^{[p]} & \longrightarrow & R_{mp+p} \\ \downarrow & & \downarrow \\ H^0(X, \mathcal{O}_X(mp)) \otimes H^0(X, \mathcal{O}_X(1))^{[p]} & \xrightarrow{f_{2,m}} & H^0(X, \mathcal{O}_X(mp+p)) \end{array}$$

where the top horizontal map is surjective for $m \geq l_1 - 1$. Now the following commutative diagram of canonical maps

$$\begin{array}{ccc} H^0(X, (F_* \mathcal{O}_X)(m)) \otimes H^0(X, \mathcal{O}_X(1)) & \longrightarrow & H^0(X, \mathcal{Q}(m)) \otimes H^0(X, \mathcal{O}_X(1)) \\ \downarrow f_{2,m} & & \downarrow \\ H^0(X, (F_* \mathcal{O}_X)(m+1)) & \xrightarrow{f_{1,m+1}} & H^0(X, \mathcal{Q}(m+1)) \end{array}$$

implies that the second vertical map is surjective, for $m \geq \tilde{m}$, as the maps $f_{2,m}$ and $f_{1,m+1}$ are surjective. This proves that \mathcal{Q} is \tilde{m} -regular. Hence the assertion (1).

(2) If

$$(2.6) \quad \chi(X, \mathcal{Q}(m)) = q_0 \binom{m+d-2}{d-2} - q_1 \binom{m+d-3}{d-3} + \dots + (-1)^{d-2} q_{d-2},$$

then by Lemma 4.1, (in the Appendix, below)

$$|q_i| \leq p^{d-1} P_i^d(\tilde{e}_0, \dots, \tilde{e}_{i+1}),$$

where $P_i^d(X_0, \dots, X_{i+1})$ is a universal polynomial function with rational coefficients.

Now, \mathcal{Q} is \tilde{m} -regular implies that, for $0 \leq i < d$, $\mathcal{Q}_i := \mathcal{Q}/(a_{\bar{d}}, \dots, a_{i+1})\mathcal{Q}$ is \tilde{m} -regular, for any \mathcal{Q} -sequence $a_1, \dots, a_{\bar{d}} \in H^0(X, \mathcal{O}_X(1))$. Therefore

$$\begin{aligned} h^0(X, \mathcal{Q}_i) &\leq h^0(X, \mathcal{Q}_i(\tilde{m})) \leq h^0(X, \mathcal{Q}_{i+1}(\tilde{m})) \leq \dots \leq h^0(X, \mathcal{Q}(\tilde{m})) = \chi(X, \mathcal{Q}(\tilde{m})) \\ &\leq |q_0| \binom{\tilde{m}+d-2}{d-2} + |q_1| \binom{\tilde{m}+d-3}{d-3} + \dots + |q_{d-2}|. \end{aligned}$$

This implies $h^0(X, \mathcal{Q}_i) \leq p^{d-1} P^d(\tilde{e}_0, \dots, \tilde{e}_{d-1}, \tilde{m})$, where $P^d(X_0, \dots, X_{d-1}, Y)$ is a universal polynomial function with rational coefficients. Therefore $C_0(\mathcal{Q}) \leq (d-1)p^{d-1} P^d(\tilde{e}_0, \dots, \tilde{e}_{d-1}, \tilde{m})$. The inequality for $D_0(\mathcal{Q})$ follows similarly. This proves the assertion (2) and hence the lemma. \square

Lemma 2.8. *Let $m_0 \geq 0$ and $n_2 \geq 0$ be two integers. Consider the short exact sequences of \mathcal{O}_X -modules*

$$0 \longrightarrow \mathcal{O}_X(-m_0) \longrightarrow \mathcal{O}_X \longrightarrow Y_1 \longrightarrow 0 \quad \text{and} \quad 0 \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{O}_X(n_2) \longrightarrow Y_2 \longrightarrow 0.$$

Then, for given d , there exist universal polynomial functions $\bar{P}_2^d(X_0, \dots, X_{d-1}, Y)$ and $\bar{P}_3^d(X_0, \dots, X_{d-1}, Y)$, with rational coefficients such that

$$2C_0(Y_1) + D_0(Y_1) \leq m_0^{d-1} \bar{P}_2^d(\tilde{e}_0, \dots, \tilde{e}_{d-1}, \bar{m}),$$

$$2C_0(Y_2) + D_0(Y_2) \leq n_2^{d-1} \bar{P}_3^d(\tilde{e}_0, \dots, \tilde{e}_{d-1}, \bar{m}).$$

Proof. Without loss of generality one can assume $m_0 \geq 1$ and $n_0 \geq 1$. Since \mathcal{O}_X is \bar{m} -regular, the sheaf Y_1 is $\bar{m} + m_0$ -regular sheaf of \mathcal{O}_X -modules of dimension $d - 2$. Therefore, for any Y_1 -sequence a_1, \dots, a_{d-2} , the sheaf $Y_{1i} := Y_1/(a_{d-2}, \dots, a_{i+1})Y_1$ is $\bar{m} + m_0$ -regular, as \mathcal{O}_X -modules. This implies

$$\begin{aligned} h^0(X, Y_{1i}) &\leq h^0(X, Y_{1i}(\bar{m} + m_0)) \leq h^0(X, Y_1(\bar{m} + m_0)) \leq h^0(X, \mathcal{O}_X(\bar{m} + m_0)) \\ &\leq m_0^{d-1} \left[|\tilde{e}_0| \binom{\bar{m} + 1 + d - 2}{d-1} + |\tilde{e}_1| \binom{m + 1 + d - 2}{d-2} + \dots + |\tilde{e}_{d-1}| \right] \\ &= m_0^{d-1} \tilde{P}^d(\tilde{e}_0, \dots, \tilde{e}_{d-1}, \bar{m}), \end{aligned}$$

where $\tilde{P}^d(X_0, \dots, X_{d-1}, Y)$ is a universal polynomial function with rational coefficients.

Let $e_i(Y)$ denote the i^{th} coefficient of the Hilbert polynomial of $(Y_1, \mathcal{O}_X(1))$. Then by Lemma 4.1, we have $e_i(Y_1) \leq m_0^{i+1} P_i^d(\tilde{e}_0, \dots, \tilde{e}_i)$, where $P_i^d(X_0, \dots, X_i)$ is a universal polynomial with rational coefficients.

Now the bound for $2C_0(Y_1) + D_0(Y_1)$ follows. The identical proof follows for Y_2 . \square

Notations 2.9. For a pair (R, I) , where R is a standard graded ring of char $p > 0$, we define (similar to the sequence we had defined in [T4]), a sequence of functions $\{f_n : [1, \infty) \rightarrow [0, \infty)\}_n$, as follows: Fix $n \in \mathbb{N}$ and denote $q = p^n$. Let $x \in \mathbb{R}$ then there exists a unique nonnegative integer m such that $(m + q)/q \leq x < (m + q + 1)/q$. We define

$$f_n(x) = 1/q^{d-1} \ell(R/I^{[q]})_{m+q}.$$

Lemma 2.10. *Each $f_n : [1, \infty) \rightarrow [0, \infty)$, defined as in Notations 2.9, is a compactly supported function such that $\cup_{n \geq 1} \text{Supp } f_n \subseteq [1, n_0\mu]$, where $R_{n_0} \subseteq I$ and $\mu = \mu(I)$.*

Proof. Since R is standard graded ring, for $m \geq n_0\mu q$, we have $R_m \subseteq (R_{n_0})^{\mu q} \subseteq I^{\mu q} \subseteq I^{[q]}$. This implies $\ell(R/I^{[q]})_m = 0$, if $m \geq n_0\mu q$. Therefore support $f_n \subseteq [1, n_0\mu]$, for every $n \geq 0$. This proves the lemma. \square

Proposition 2.11. *For f_n as given in Notations 2.9, we have*

- (1) $|f_n(x) - f_{n+1}(x)| \leq C/p^{n-d+2}$, for every $x \in [1, \infty)$, and for all $n \geq 0$.
- (2) In particular, $\|f_n - f_{n+1}\| \leq C/p^{n-d+2}$ and $\|f_{d-1} - f_d\| \leq C/p$,

where

$$(2.7) \quad C = C_R + \mu \left(\bar{m} + n_0 \left(\sum_{i=1}^{\mu} d_i \right) + 1 \right)^{d-2} (\bar{P}_1^d + d^{d-1} \bar{P}_2^d + \bar{P}_3^d)$$

and the integers \bar{m} and n_0 are given as in Notations 2.2, and d_1, \dots, d_μ are degrees of a chosen generators of I . Moreover $C_R = \mu h^0(X, \mathcal{O}_X(\bar{m}))$, for $X = \text{Proj } R$, and \bar{P}_1^d , \bar{P}_2^d and \bar{P}_3^d are given as in Lemma 2.7 and Lemma 2.8 respectively.

Proof. Fix $x \in [1, \infty)$. Therefore, for given $q = p^n$, there exists a unique integer $m \geq 0$, such that $(m + q)/q \leq x < (m + q + 1)/q$ and

$$\frac{(m + q)p + n_2}{qp} \leq x < \frac{(m + q)p + n_2 + 1}{qp}, \text{ for some } 0 \leq n_2 < p.$$

Hence

$$f_n(x) = \frac{1}{q^{d-1}} \ell \left(\frac{R}{I^{[q]}} \right)_{m+q} \quad \text{and} \quad f_{n+1}(x) = \frac{1}{(qp)^{d-1}} \ell \left(\frac{R}{I^{[qp]}} \right)_{mp+qp+n_2}.$$

Now, by Equation (2.3) in Remark 2.3, we have

$$(2.8) \quad \left| f_n(x) - \frac{\text{coker } \phi_{m,q}(\mathcal{O}_X)}{q^{d-1}} \right| < \frac{C_R}{q^{d-1}} \quad \text{and} \quad \left| f_{n+1}(x) - \frac{\text{coker } \phi_{mp+n_2,qp}(\mathcal{O}_X)}{(qp)^{d-1}} \right| < \frac{C_R}{(qp)^{d-1}}.$$

Consider the short exact sequence of \mathcal{O}_X -modules

$$0 \longrightarrow \mathcal{O}_X(-d) \longrightarrow \mathcal{O}_X \longrightarrow Y_1 \longrightarrow 0.$$

Then, for any locally free sheaf P of \mathcal{O}_X -modules and for $m \geq 0$, we have the following short exact sequence of \mathcal{O}_X -modules

$$0 \longrightarrow F^{n*}P \otimes \mathcal{O}_X(-d+m) \longrightarrow F^{n*}P \otimes \mathcal{O}_X(m) \longrightarrow F^{n*}P \otimes Y_1(m) \longrightarrow 0.$$

Since

$$\text{coker } \phi_{m,q}(\mathcal{O}_X) = h^0(X, \mathcal{O}_X(m+q)) - \sum_i h^0(X, \mathcal{O}_X(q - qd_i)) + h^0(X, (F^{n*}V)(m))$$

we have (by taking $P = V$ and $= \sum \mathcal{O}_X(1 - d_i)$) respectively),

$$\begin{aligned} & |\text{coker } \phi_{m,q}(\mathcal{O}_X) - \text{coker } \phi_{m-d,q}(\mathcal{O}_X)| \\ & \leq h^0(X, Y_1(m+q)) + h^0(X, \sum_i \mathcal{O}_X(q - qd_i) \otimes Y_1(m)) + h^0(X, F^{n*}V \otimes Y_1(m)) \end{aligned}$$

which, by Lemma 2.5 is

$$\leq (\mu)D_0(Y_1)(m+q)^{d-2} + 2(\mu)C_0(Y_1)(m^{d-2} + 1) \leq (\mu)[2C_0(Y_1) + D_0(Y_1)](m+q)^{d-2}.$$

Therefore

$$(2.9) \quad |p^{d-1} \text{coker } \phi_{m,q}(\mathcal{O}_X) - p^{d-1} \text{coker } \phi_{m-d,q}(\mathcal{O}_X)| \leq (\mu)[2C_0(Y_1) + D_0(Y_1)](m+q)^{d-2}p^{d-1}.$$

Since, for a locally free sheaf P , we have

$$h^0(X, F^{n*}P \otimes (F_*\mathcal{O}_X)(m)) = h^0(X, F^{(n+1)*}P \otimes \mathcal{O}_X(mp)),$$

the short exact sequence in the statement of Lemma 2.7 gives a canonical long exact sequence

$$0 \longrightarrow H^0(X, (F^{n*}P)(m-d)) \longrightarrow H^0(X, (F^{(n+1)*}P)(mp)) \longrightarrow H^0(X, \mathcal{Q}(m)) \longrightarrow \cdots,$$

which implies

$$(2.10) \quad |p^{d-1} \text{coker } \phi_{(m-d),q}(\mathcal{O}_X) - \text{coker } \phi_{mp,qp}(\mathcal{O}_X)| \leq (\mu)[2C_0(\mathcal{Q}) + D_0(\mathcal{Q})](m+q)^{d-2}.$$

The short exact sequence of \mathcal{O}_X -modules

$$0 \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{O}_X(n_2) \longrightarrow Y_2 \longrightarrow 0$$

gives

$$0 \longrightarrow H^0(X, (F^{(n+1)*}P)(mp)) \longrightarrow H^0(X, (F^{(n+1)*}P)(mp+n_2)) \longrightarrow H^0(X, (F^{(n+1)*}P) \otimes Y_2(mp)),$$

which gives

$$\begin{aligned} & |\text{coker } \phi_{mp,qp}(\mathcal{O}_X) - \text{coker } \phi_{mp+n_2,qp}(\mathcal{O}_X)| \\ & \leq h^0(X, F^{(n+1)*}V \otimes Y_2(mp)) + h^0(X, \sum_i \mathcal{O}_X(qp - qpd_i) \otimes Y_2(mp)) + h^0(X, Y_2(mp+qp)) \\ & \leq 2(\mu)C_0(Y_2)((mp)^{d-2} + 1) + (\mu)D_0(Y_2)(mp+qp)^{d-2}. \end{aligned}$$

Therefore

$$(2.11) \quad |\text{coker } \phi_{mp,qp}(\mathcal{O}_X) - \text{coker } \phi_{mp+n_2,qp}(\mathcal{O}_X)| \leq (\mu)[2C_0(Y_2) + D_0(Y_2)](mp+qp)^{d-2}.$$

Combining Equations (2.9), (2.10) and (2.11), we get

$$\begin{aligned} (A) &:= |p^{d-1} \operatorname{coker} \phi_{m,q}(\mathcal{O}_X) - \operatorname{coker} \phi_{mp+n_2,qp}(\mathcal{O}_X)| \\ &\leq (\mu)(m+q)^{d-2} [(2C_0(Y_1) + D_0(Y_1))p^{d-1} + (2C_0(\mathcal{Q}) + D_0(\mathcal{Q})) + (2C_0(Y_2) + D_0(Y_2))p^{d-2}] \\ &\leq (\mu)(m+q)^{d-2} [p^{d-1}d^{d-1}\bar{P}_2^d + p^{d-1}\bar{P}_1^d + n_2^{d-1}\bar{P}_3^d p^{d-2}], \end{aligned}$$

where the last inequality follows from Lemma 2.7, Lemma 2.8. Now, as $n_2 < p$, we have

$$(A) \leq (\mu)(m+q)^{d-2} [p^{d-1}d^{d-1}\bar{P}_2^d + p^{d-1}\bar{P}_1^d + p^{d-1}p^{d-2}\bar{P}_3^d].$$

Now multiplying the above inequality by $1/(qp)^{d-1}$ we get

$$\left| \frac{\operatorname{coker} \phi_{m,q}(\mathcal{O}_X)}{q^{d-1}} - \frac{\operatorname{coker} \phi_{mp+n_2,qp}(\mathcal{O}_X)}{(qp)^{d-1}} \right| \leq (\mu) \frac{(m+q)^{d-2}}{q^{d-1}} [d^{d-1}\bar{P}_2^d + \bar{P}_1^d + p^{d-2}\bar{P}_3^d].$$

Moreover, by Remark 2.3,

$$m+q \geq \bar{m} + n_0 \left(\sum_i d_i \right) q + q \implies \operatorname{coker} \phi_{m,q}(\mathcal{O}_X) = \operatorname{coker} \phi_{mp+n_2,qp}(\mathcal{O}_X) = 0.$$

Also

$$m+q \leq \bar{m} + n_0 \left(\sum_i d_i \right) q + q, \implies (m+q)^{d-2} \leq L_0 q^{d-2}, \text{ where } L_0 = (\bar{m} + n_0 \left(\sum_i d_i \right) + 1)^{d-2}.$$

Therefore, for every $m \geq 0$ and $n \geq 1$, where $q = p^n$, we have

$$\begin{aligned} \left| \frac{\operatorname{coker} \phi_{m,q}(\mathcal{O}_X)}{q^{d-1}} - \frac{\operatorname{coker} \phi_{mp+n_2,qp}(\mathcal{O}_X)}{(qp)^{d-1}} \right| &\leq (\mu) \frac{L_0 q^{d-2}}{q^{d-1}} [d^{d-1}\bar{P}_2^d + \bar{P}_1^d + p^{d-2}\bar{P}_3^d] \\ &\leq (\mu) L_0 [d^{d-1}\bar{P}_2^d + \bar{P}_1^d + \bar{P}_3^d] \frac{p^{d-2} q^{d-2}}{q^{d-1}}. \end{aligned}$$

Now by Equation 2.8, we have

$$|f_n(x) - f_{n+1}(x)| \leq \frac{C_R}{q^{d-1}} + \frac{C_R}{(qp)^{d-1}} + (\mu) L_0 [d^{d-1}\bar{P}_2^d + \bar{P}_1^d + \bar{P}_3^d] \frac{p^{d-2}}{q} \leq C \frac{p^{d-2}}{q},$$

where $C = C_R + (\mu) L_0 (d^{d-1}\bar{P}_2^d + \bar{P}_1^d + \bar{P}_3^d)$, which proves the proposition. \square

Corollary 2.12. *There exists a constant $C_1 = P_4^d(\tilde{e}_0, \tilde{e}_1, \dots, \tilde{e}_d, \bar{m}) + (n_0\mu - 1)C$, where C is as in Proposition 2.11 and $P_4^d(X_0, \dots, X_d, Y)$ is a universal polynomial function with rational coefficients such that, for $n \geq 1$*

$$\left| \frac{1}{(p^n)^d} \ell \left(\frac{R}{I[p^n]} \right) - \frac{1}{(p^{n+1})^d} \ell \left(\frac{R}{I[p^{n+1}]} \right) \right| \leq \frac{C_1}{p^{n-d+2}}.$$

Proof. Note

$$\begin{aligned} &\left| \frac{1}{(p^n)^d} \ell \left(\frac{R}{I[p^n]} \right) - \frac{1}{(p^{n+1})^d} \ell \left(\frac{R}{I[p^{n+1}]} \right) \right| \\ &\leq \left| \frac{1}{p^{nd}} \ell \left(\frac{R}{\mathbf{m}^{p^n}} \right) - \frac{1}{p^{(n+1)d}} \ell \left(\frac{R}{\mathbf{m}^{p^{n+1}}} \right) \right| + \left| \int_1^{n_0\mu} f_n(x) dx - \int_1^{n_0\mu} f_{n+1}(x) dx \right|. \end{aligned}$$

If $p \leq \bar{m}$ then

$$\left| \frac{1}{p^{nd}} \ell \left(\frac{R}{\mathbf{m}^{p^n}} \right) - \frac{1}{p^{(n+1)d}} \ell \left(\frac{R}{\mathbf{m}^{p^{n+1}}} \right) \right| \leq \left| \frac{P_{(R,\mathbf{m})}(\bar{m})}{p^{nd}} - \frac{P_{(R,\mathbf{m})}(\bar{m}^2)}{p^{(n+1)d}} \right| \leq \frac{P_{(R,\mathbf{m})}(\bar{m}^2)}{p^n},$$

if $p \geq \bar{m}$ then there exists a universal polynomial function $P_6^d(X_0, \dots, X_d)$ with rational coefficients such that

$$\text{L.H.S.} \leq \left| \frac{P_{(R,\mathbf{m})}(p^n)}{p^{nd}} - \frac{P_{(R,\mathbf{m})}(p^{n+1})}{p^{(n+1)d}} \right| \leq \frac{P_6^d(\tilde{e}_0, \tilde{e}_1, \dots, \tilde{e}_d)}{p^n}.$$

Therefore there exists a universal polynomial function $P_4^d(X_0, \dots, X_d, Y)$ with rational coefficients such that

$$\left| \frac{1}{(p^n)^d} \ell \left(\frac{R}{I[p^n]} \right) - \frac{1}{(p^{n+1})^d} \ell \left(\frac{R}{I[p^{n+1}]} \right) \right| \leq \frac{P_4^d(\tilde{e}_0, \tilde{e}_1, \dots, \tilde{e}_d)}{p^n} + \frac{(n_0\mu - 1)C}{p^{n-d+2}}.$$

Since $d \geq 2$, the corollary follows. \square

3. HILBERT-KUNZ DENSITY FUNCTION AND REDUCTION MOD p

Remark 3.1. Let R be a standard graded integral domain of dimension $d \geq 2$, with $R_0 = k$, where k is an algebraically closed field. Let $N = \ell(R_1)$, then we have a surjective graded map $k[X_0, \dots, X_N] \rightarrow R$ of degree 0, given by X_i mapping to generators of R_1 . This gives a closed immersion $X = \text{Proj } R \rightarrow \mathbb{P}_k^N$ such that $\mathcal{O}_X(1) = \mathcal{O}_{\mathbb{P}_k^N}(1)|_X$. Therefore if

$$P_R(m) = \tilde{e}_0 \binom{m+d-1}{d} - \tilde{e}_1 \binom{m+d-2}{d-1} + \dots + (-1)^d \tilde{e}_d$$

is the Hilbert-Samuel polynomial of R then the Hilbert polynomial for the pair $(X, \mathcal{O}_X(1))$ is

$$\chi(X, \mathcal{O}_X(m)) = \tilde{e}_0 \binom{m+d-1}{d-1} - \tilde{e}_1 \binom{m+d-2}{d-2} + \dots + (-1)^{d-1} \tilde{e}_{d-1}.$$

Since R is a domain, the canonical graded map $R = \oplus_m R_m \rightarrow \oplus_m H^0(X, \mathcal{O}_X(m))$ is injective.

Let \mathcal{I}_X be the ideal sheaf of X in \mathbb{P}_k^N then we have the canonical short exact sequence of $\mathcal{O}_{\mathbb{P}_k^N}$ -modules

$$0 \rightarrow \mathcal{I}_X \rightarrow \mathcal{O}_{\mathbb{P}_k^N} \rightarrow \mathcal{O}_X \rightarrow 0$$

and the image of the induced map $f_m : H^0(\mathbb{P}_k^N, \mathcal{O}_{\mathbb{P}_k^N}(m)) \rightarrow H^0(X, \mathcal{O}_X(m))$ is R_m . Now, by Exp XIII, (6.2) (in [SGA6]), there exists a universal polynomial $P_5^d(X_0, \dots, X_{d-1})$ with rational coefficients such that the sheaf \mathcal{I}_X is $\tilde{m} = P_5^d(\tilde{e}_0, \dots, \tilde{e}_{d-1})$ -regular. Therefore the map f_m is surjective for $m \geq \tilde{m}$.

In particular, we have

- (1) $R_m = H^0(X, \mathcal{O}_X(m))$, for all $m \geq \tilde{m}$ and
- (2) the sheaf \mathcal{O}_X is \tilde{m} -regular with respect to $\mathcal{O}_X(1)$.

Here we recall a notion of spread.

Definition 3.2. Consider the pair (R, I) , where R is a finitely generated \mathbb{N} -graded d dimensional domain over an algebraically closed field k of characteristic 0 and $I \subset R$ is a homogeneous ideal of finite colength. For such a pair, there exists a finitely generated \mathbb{Z} -algebra $A \subseteq k$, a finitely generated \mathbb{N} -graded algebra R_A over A and a homogeneous ideal $I_A \subset R_A$ such that $R_A \otimes_A k = R$ and $I = \text{Im}(I_A \otimes_A k)$. We call (A, R_A, I_A) a *spread* of the pair (R, I) .

Moreover, if, for the pair (R, I) , we have a spread (A, R_A, I_A) as above and $A \subset A' \subset k$, for some finitely generated \mathbb{Z} -algebra A' then $(A', R_{A'}, I_{A'})$ satisfy the same properties as (A, R_A, I_A) . Hence we may always assume that the spread (A, R_A, I_A) as above is chosen such that A contains a given finitely generated \mathbb{Z} -algebra $A_0 \subseteq k$.

Remark 3.3. Note that for a spread (A, R_A, I_A) of (R, I) as above, the induced map

$\tilde{\pi} : X_A := \text{Proj } R_A \rightarrow \text{Spec}(A)$ is a proper map, hence by generic flatness there is an open set (infact non empty as A is an integral domain) $U \subset \text{Spec}(A)$ such that $\tilde{\pi}|_{\tilde{\pi}^{-1}(U)} : \tilde{\pi}^{-1}(U) \rightarrow U$ is a proper flat map. Therefore (see [EGA IV] 12.2.1) the set

$$\{s \in \text{Spec}(A) \mid X \otimes_{\text{Spec}(A)} \text{Spec}(k(s)) \text{ is geometrically integral}\}$$

is a nonempty open set of $\text{Spec}(A)$. Hence replacing A by a finitely generated \mathbb{Z} -algebra A' such that $A \subset A' \subset k$ (if necessary) we can assume that $\tilde{\pi}$ is a flat map such that for every $s \in \text{Spec}(A)$, the fiber over s is geometrically integral.

Therefore for any closed point $s \in \text{Spec}(A)$ (i.e., a maximal ideal of A) the ring $R_s = R_A \otimes_A k(s)$ is a standard graded d -dimensional ring such that the ideal $I_s = \text{Im}(I_A \otimes_A k(s)) \subset R_s$ is a homogeneous ideal of finite colength. $X_s := X \otimes k(s)$ is integral scheme over $k(s)$.

PROOF of Theorem 1.1 For given (R, I) , and a given spread (A, R_A, I_A) , we can choose a spread $(A', R_{A'}, I_{A'})$, where $A \subset A'$, such that the induced projective morphism of Noetherian schemes $\tilde{\pi} : X_{A'} \rightarrow A'$ is flat and, for every $s \in \text{Spec}(A')$, X_s is an integral scheme over $\bar{k}(s)$ of dimension $= d - 1$. Let $R_{A'} = \bigoplus_{i \geq 0} (R_{A'})_i$ and let $(R_{A'})_1$ be generated by N elements as an A' -module. Then the canonical graded surjective map

$$A'[X_0, \dots, X_N] \rightarrow R_{A'},$$

gives a closed immersion $X_{A'} = \text{Proj } R_{A'} \rightarrow \mathbb{P}_{A'}^N$ such that $\mathcal{O}_{X_{A'}}(1) = \mathcal{O}_{\mathbb{P}_{A'}^N}(1)|_{X_{A'}}$. Let $X_s = X_{A'} \otimes \bar{k}(s)$. Then $X_s = \text{Proj } R_s$ and $\mathcal{O}_{X_s}(1)$ is the canonical line bundle induced by $\mathcal{O}_{X_{A'}}(1)$. Let $s_0 = \text{Spec } Q(A) = \text{Spec } Q(A')$ be the generic point of $\text{Spec}(A')$. We now have the following,

- (1) The Hilbert polynomial for the pair $(X_s, \mathcal{O}_{X_s}(1))$ is

$$\chi(X_s, \mathcal{O}_{X_s}(m)) = \tilde{e}_0 \binom{m+d-1}{d-1} - \tilde{e}_1 \binom{m+d-2}{d-2} + \dots + (-1)^{d-1} \tilde{e}_{d-1},$$

where the coefficients \tilde{e}_i are as above for $(X, \mathcal{O}_X(1))$ (from char 0).

In particular, $\dim X_s = d - 1$ and

- (2) by Remark 3.1, there exists $\bar{m} = P_5^d(\tilde{e}_0, \dots, \tilde{e}_{d-1})$ such that $(R_s)_m = H^0(X_s, \mathcal{O}_{X_s}(m))$ for all $m \geq \bar{m}$ and $(X_s, \mathcal{O}_{X_s}(1))$ is \bar{m} -regular.
(3) Moreover, by the semicontinuity theorem (Chapter III, Theorem 12.8 in [H]), by shrinking $\text{Spec}(A')$ further, we have $h^i(X_s, \mathcal{O}_{X_s}(\bar{m}))$ and $h^0(X_s, \mathcal{O}_{X_s})$ is independent of s , for all $i \geq 0$.
(4) Again by shrinking $\text{Spec}(A')$ (if necessary), can choose $n_0 \in \mathbb{N}$ such that $R_{A'}^{n_0} \subseteq I_{A'}$. This implies $R_s^{n_0} \subseteq I_s$.

Let us fix a $s \in \text{Spec}(A')$. We sketch the proof of the existence of the map $f^s : [1, \infty) \rightarrow \mathbb{R}$ and its relation to $e_{HK}(R, I)$ (note that we have proved this in a more general setting in [T4]). By Proposition 2.11, for any given s , the sequence $\{f_n^s\}_n$ of functions is uniformly convergent. Let $f^s(x) = \lim_{n \rightarrow \infty} f_n^s(x)$. This implies that $\lim_{n \rightarrow \infty} \int_1^\infty f_n^s(x) = \int_1^\infty f^s(x)$, as, by Lemma 2.10, there is a compact set containing $\cup_n \text{supp } f_n^s$. On the other hand

$$\begin{aligned} e_{HK}(R_s, I_s) &= \lim_{n \rightarrow \infty} \frac{1}{p^{nd}} \ell \left(\frac{R_s}{I_s^{[p^n]}} \right) = \lim_{n \rightarrow \infty} \frac{1}{p^{nd}} \ell \left(\frac{R_s}{\mathbf{m}_s^{p^n}} \right) + \lim_{n \rightarrow \infty} \frac{1}{p^{nd}} \sum_{m \geq 0} \ell \left(\frac{R}{I^{[p^n]}} \right)_{m+p^n} \\ &= e(R_s, \mathbf{m}_s) + \lim_{n \rightarrow \infty} \int_1^\infty f_n^s(x) = e(R_s, \mathbf{m}_s) + \int_1^\infty f^s(x), \end{aligned}$$

where \mathbf{m}_s is the graded maximal ideal of R_s . Now, by Proposition 2.11, there exists a constant

$$C = C_{R_s} + \mu \left(\bar{m} + n_0 \left(\sum_{i=1}^{\mu} d_i \right) + 1 \right)^{d-2} (\bar{P}_1^d + d^{d-1} \bar{P}_2^d + \bar{P}_3^d),$$

which is independent of s (as $C_{R_s} = \mu h^0(X_s, \mathcal{O}_{X_s}(1))$) such that

$$\|f_n^s - f_{n+1}^s\| \leq C/p^{n-d+2}, \quad \text{for all } n.$$

In particular, for given $m \geq d - 1$,

$$\|f_m^s - f^s\| \leq [C/p + C/p^2 + C/p^3 + \dots] \frac{1}{p^{m-(d-1)}} \leq \frac{2C}{p^{m-d+2}}.$$

As $s \rightarrow s_0$ we have $\text{char } k(s) \rightarrow \infty$, which implies $\lim_{s \rightarrow s_0} \|f_m^s - f^s\| = 0$. This proves the first assertion of the theorem.

Since each f_m^s and f^s has support in the compact interval $[1, n_0\mu]$, the above inequality implies that, for any fixed $m \geq d - 1$,

$$\lim_{s \rightarrow s_0} \left| \int_1^\infty f_m^s(x) dx - \int_1^\infty f^s(x) dx \right| \leq \lim_{s \rightarrow s_0} \int_1^\infty |f_m^s(x) - f^s(x)| dx \leq \lim_{s \rightarrow s_0} \left(\frac{2C}{p^{m-d+2}} \right) (n_0\mu - 1) = 0.$$

Moreover it is easy to see that

$$\lim_{s \rightarrow s_0} \left[\frac{1}{p^{(m)d}} \ell \left(\frac{R_s}{\mathbf{m}_s^{p^m}} \right) - e(R_s, \mathbf{m}_s) \right] = 0.$$

Therefore

$$\begin{aligned} & \lim_{s \rightarrow s_0} \left[\frac{1}{p^{(m)d}} \ell \left(\frac{R_s}{I_s^{[p^m]}} \right) - e_{HK}(R_s, I_s) \right] = \\ & \lim_{s \rightarrow s_0} \left[\left\{ \frac{1}{p^{md}} \ell \left(\frac{R_s}{\mathbf{m}_s^{p^m}} \right) + \int_1^\infty f_m^s(x) dx \right\} - \left\{ e(R_s, \mathbf{m}_s) + \int_1^\infty f^s(x) dx \right\} \right] = 0. \end{aligned}$$

□

Now the proof of Corollary 1.2 is obvious.

4. APPENDIX

Lemma 4.1. *Let the Hilbert polynomial of $(X, \mathcal{O}_X(1))$ be given by*

$$\chi(X, \mathcal{O}_X(m)) = \tilde{e}_0 \binom{m+d-1}{d-1} - \tilde{e}_1 \binom{m+d-2}{d-2} + \cdots + (-1)^{d-1} \tilde{e}_{d-1}.$$

Let

$$\chi(X, \mathcal{Q}(m)) = q_0 \binom{m+d-2}{d-2} - q_1 \binom{m+d-3}{d-3} + \cdots + (-1)^{d-2} q_{d-2},$$

be the Hilbert polynomial of $(\mathcal{Q}, \mathcal{O}_X(1))$ where \mathcal{Q} is given by one of the following short exact sequences.

(1) Let

$$0 \longrightarrow \oplus^{p^{d-1}} \mathcal{O}_X(-d) \longrightarrow F_* \mathcal{O}_X \longrightarrow \mathcal{Q} \longrightarrow 0$$

as in the Lemma 2.6. Then, for each given $0 \leq i \leq d-2$, there exists a universal polynomial $P_i^d = P^d(X_0, \dots, X_i)$ with rational coefficients such that

$$|q_i| \leq p^{d-1} P^d(\tilde{e}_0, \dots, \tilde{e}_{i+1}).$$

(2) Let

$$0 \longrightarrow \mathcal{O}_X(-m_0) \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{Q} \longrightarrow 0$$

or

$$0 \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{O}_X(m_0) \longrightarrow \mathcal{Q} \longrightarrow 0.$$

Then for $0 \leq i \leq d-2$, there exists a universal polynomial $P_i'^d = P'^d(X_0, \dots, X_i)$ with rational coefficients such that

$$|q_i| \leq m_0^{i+1} P'^d(\tilde{e}_0, \dots, \tilde{e}_i).$$

Proof. Assertion (1): Note that for $m \in \mathbb{Z}$, we have

$$(4.1) \quad \chi(X, \mathcal{Q}(m)) = \chi(X, \mathcal{O}_X(mp)) - p^{d-1} \chi(X, \mathcal{O}_X(m)).$$

We can express, for $1 \leq d-i \leq d-1$,

$$(Y+n) \cdots (Y+2)(Y+1) = \sum_{j=0}^n C_j^n Y^j,$$

where $C_n^n = 1$ and, for $j < n$,

$$C_j^n \in \left\{ \sum x_1^{i_1} \cdots x_n^{i_n} \mid \sum i_l = n-j, \quad 0 \leq j < n, \quad \{x_1, \dots, x_n\} = \{1, \dots, n\} \right\}.$$

Now expanding Equation (4.1), we get

$$\begin{aligned} & \frac{\tilde{e}_0}{(d-1)!} [C_{d-2}^{d-1} m^{d-2} (p^{d-2} - p^{d-1}) + C_{d-3}^{d-1} m^{d-3} (p^{d-3} - p^{d-1}) + \cdots + C_0^{d-1} (1 - p^{d-1})] \\ & + \cdots + \frac{(-1)^i \tilde{e}_i}{(d-1-i)!} [C_{d-1-i}^{d-1-i} m^{d-1-i} (p^{d-1-i} - p^{d-1}) + C_{d-2-i}^{d-1-i} m^{d-2-i} (p^{d-2-i} - p^{d-1})] \end{aligned}$$

$$\begin{aligned}
& + \cdots + C_0^{d-1-i}(1-p^{d-1})] + \cdots + (-1)^{d-1}\tilde{e}_{d-1}[(1-p^{d-1})] \\
& = \frac{q_0}{(d-2)!} [C_{d-2}^{d-2}m^{d-2} + C_{d-3}^{d-2}m^{d-3} + \cdots + C_0^{d-2}] - \frac{q_1}{(d-3)!} [C_{d-3}^{d-3}m^{d-3} + C_{d-4}^{d-3}m^{d-4} + \cdots + C_0^{d-3}] \\
& + \cdots + \frac{(-1)^{i-1}q_{i-1}}{(d-1-i)!} [C_{d-1-i}^{d-1-i}m^{d-1-i} + C_{d-2-i}^{d-1-i}m^{d-2-i} + \cdots + \cdots + C_0^{d-1-i}] + \cdots + (-1)^{d-2}q_{d-2}.
\end{aligned}$$

We prove the result for q_i , by induction on i . For $i = 0$, comparing the coefficients of m^{d-2} on both the sides we get

$$(p^{d-2} - p^{d-1}) \left[\frac{\tilde{e}_0}{(d-1)!} C_{d-2}^{d-1} - \frac{\tilde{e}_1}{(d-2)!} \right] = \frac{q_0}{(d-2)!} \implies |q_0| \leq p^{d-1} (|\tilde{e}_0| C_{d-2}^{d-1} + |\tilde{e}_1|).$$

Comparing coefficients of m^{d-i} we get

$$\begin{aligned}
& (p^{d-i} - p^{d-1}) \left[\frac{\tilde{e}_0}{(d-1)!} C_{d-i}^{d-1} - \frac{\tilde{e}_1}{(d-2)!} C_{d-i}^{d-2} + \cdots + (-1)^{i-1} \frac{\tilde{e}_{i-1}}{(d-i)!} C_{d-i}^{d-1-i} \right] \\
& = \frac{q_0}{(d-2)!} C_{d-i}^{d-2} - \frac{q_1}{(d-3)!} C_{d-i}^{d-3} + \cdots + (-1)^i \frac{q_{i-2}}{(d-i)!} C_{d-i}^{d-i}.
\end{aligned}$$

This implies that

$$|q_{i-2}| \leq p^{d-1} [|\tilde{e}_0| C_{d-i}^{d-1} + |\tilde{e}_1| C_{d-i}^{d-2} + \cdots + |\tilde{e}_{i-1}| C_{d-i}^{d-1-i}] + [|q_0| C_{d-i}^{d-2} + |q_1| C_{d-i}^{d-3} + \cdots + |q_{i-3}| C_{d-i}^{d+1-i}].$$

Now the proof follows by induction.

Assertion (2): For $m_0 = 0$ the statement is true vacuously. Therefore we can assume that $m_0 \geq 1$. Now

$$\begin{aligned}
\chi(X, \mathcal{Q}(m)) & = q_0 \binom{m+d-2}{d-2} - q_1 \binom{m+d-3}{d-3} + \cdots + (-1)^{d-2} q_{d-2}, \\
& = \frac{q_0}{(d-2)!} [D_{d-2}^{d-2}m^{d-2} + D_{d-3}^{d-2}m^{d-3} + \cdots + D_0^{d-2}] - \frac{q_1}{(d-3)!} [D_{d-3}^{d-3}m^{d-3} + D_{d-4}^{d-3}m^{d-4} + \cdots + D_0^{d-3}] \\
& + \cdots + \frac{(-1)^{i-1}q_{i-1}}{(d-1-i)!} [D_{d-1-i}^{d-1-i}m^{d-1-i} + D_{d-2-i}^{d-1-i}m^{d-2-i} + \cdots + \cdots + D_0^{d-1-i}] + \cdots + (-1)^{d-2} q_{d-2},
\end{aligned}$$

where

$$D_j^k = \left\{ \sum x^{i_1} \cdots x_k^{i_k} \mid \sum i_l = k - j \quad 0 \leq j \leq k \leq d-1, \quad \{x_1, \dots, x_k\} = \{1, \dots, k\} \right\}.$$

On the other hand

$$\begin{aligned}
\chi(X, \mathcal{O}_X(m)) - \chi(X, \mathcal{O}_X(m - m_0)) & = \frac{\tilde{e}_0}{(d-1)!} [C_{d-1}^{d-1}(m_0)(m^{d-2} + \cdots m^{d-3}m_0 + \cdots + m_0^{d-2}) \\
& + C_{d-2}^{d-1}(m_0)(m^{d-3} + m^{d-4}m_0 + \cdots + m_0^{d-3}) + \cdots + C_1^{d-1}(m_0)] \\
& - \frac{\tilde{e}_1}{(d-2)!} [C_{d-2}^{d-2}(m_0)(m^{d-3} + \cdots m^{d-4}m_0 + \cdots + m_0^{d-3}) \\
& + C_{d-3}^{d-2}(m_0)(m^{d-4} + m^{d-5}m_0 + \cdots + m_0^{d-4}) + \cdots + C_1^{d-2}(m_0)] + \cdots.
\end{aligned}$$

Again we prove the result for q_i , by induction on i . Comparing the coefficients for m^{d-2} we get

$$\frac{q_0}{(d-2)!} D_{d-2}^{d-2} = \frac{\tilde{e}_0}{(d-1)!} C_{d-1}^{d-1} m_0 \implies |q_0| \leq \tilde{e}_0 \frac{C_{d-1}^{d-1} m_0}{|D_{d-2}^{d-2}|}$$

Comparing coefficients of m^{d-i} , where $2 \leq i \leq d$, we get

$$\begin{aligned}
& \frac{q_0}{(d-2)!} D_{d-i}^{d-2} - \frac{q_1}{(d-3)!} D_{d-i}^{d-3} + \cdots + (-1)^{i-2} \frac{q_{i-2}}{(d-i)!} D_{d-i}^{d-i} \\
& = \frac{\tilde{e}_0}{(d-1)!} (C_{d-1}^{d-1} m_0^{i-1} + C_{d-2}^{d-1} m_0^{i-2} + \cdots + C_{d-i+1}^{d-1} m_0) \\
& - \frac{\tilde{e}_1}{(d-2)!} (C_{d-2}^{d-2} m_0^{i-2} + C_{d-3}^{d-2} m_0^{i-3} + \cdots + C_{d-i+1}^{d-2} m_0) + \cdots + (-1)^{i-2} \frac{\tilde{e}_{i-2}}{(d+1-i)!} (C_{d+1-i}^{d+1-i}).
\end{aligned}$$

This implies that

$$|q_{i-2}||D_{d-i}^{d-i}| \leq |\tilde{e}_0| (C_{d-1}^{d-1}m_0^{i-1} + \cdots + C_{d-i+1}^{d-1}m_0) + |\tilde{e}_1| (C_{d-2}^{d-2}m_0^{i-2} + \cdots + C_{d-i+1}^{d-2}m_0) \\ + \cdots + |\tilde{e}_{i-2}| (C_{d+1-i}^{d+1-i}) + (|q_0||D_{d-i}^{d-2}| + |q_1||D_{d-i}^{d-3}| + \cdots + |q_{i-3}||D_{d-i}^{d+1-i}|).$$

Now the proof follows by induction.

For \mathcal{Q} such that

$$0 \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{O}_X(m_0) \longrightarrow \mathcal{Q} \longrightarrow 0,$$

we have $\chi(X, \mathcal{Q}(m - m_0)) = \chi(X, \mathcal{O}_X(m)) - \chi(X, \mathcal{O}(m - m_0))$, so we get same bound for q'_i s in terms of \tilde{e}'_j s as above except that now

$$D_j^n \in \left\{ \sum x_1^{i_1} \cdots x_n^{i_n} \mid \sum_{l=1}^n i_l = n - j, \quad 0 \leq j \leq n, \quad \{x_1, \dots, x_n\} = \{1 - m_0, \dots, n - m_0\} \right\}.$$

Hence the lemma follows. \square

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